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Massless Boundary Sine-Gordon at the Free Fermion Point: Correlation and Partition Functions with Applications to Quantum Wires

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In this report we compute the boundary states (including the boundary entropy) for the boundary sine-Gordon theory. From the boundary states, we derive both correlation and partition functions. Through the partition function, we show that boundary sine-Gordon maps onto a doubled boundary Ising model. With the current-current correlators, we calculate for finite system size the ac-conductance of tunneling quantum wires with dimensionless free conductance 1/2 (or, alternatively interacting quantum Hall edges at filling fraction $\nu = 1/2$). In the dc limit, the results of [1] are reproduced.

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1. Introduction

The massless sine-Gordon theory with an integrable boundary perturbation provides a theoretical realization for a variety of statistical mechanical systems. Among those that have attracted the most attention are interacting quantum Hall edges [2] [3] and tunneling in quantum wires [1][4]. At present our understanding of the boundary sine-Gordon theory at arbitrary coupling is limited to knowledge of the boundary scattering matrices. In [5] these scattering matrices were derived from the imposition of the boundary Yang-Baxter equation and the crossing-unitarity condition. On one occasion, knowledge of these matrices was sufficient for the calculation of a physical parameter in these systems. In [3], by cleverly coupling thermodynamic Bethe ansatz (TBA) techniques to a Boltzmann transport equation, the conductance of interacting quantum Hall edges at filling fraction $\nu = 1/3$ was computed. However, in general, knowledge of correlation functions is needed to access physical quantities. But presently the general form of correlation functions is unknown.

The computation of correlation functions is facilitated by knowledge of boundary states $|B\rangle$. In two dimensional boundary field theory there are two possible pictures in which to work: one with the boundary is in time, as an initial condition, and one with boundary is in space. It is in the former picture that the boundary state is used to calculate correlation functions. Such functions then have the general form:

$$\langle O_1(x_1)\cdots O_n(x_n)\rangle = \frac{\langle 0|O_1(x_1)\cdots O_n(x_n)|B\rangle}{\langle 0|B\rangle}. \tag{1.1}$$

As the boundary is in time, the Hilbert space of the theory remains unchanged from its bulk counterpart. As such, the boundary state $|B\rangle$ is expressible in terms of these original states.

Integrability imposes powerful constraints on the form this expression must take. If $\{A_a(\theta)\}_{a\in A}$ is a particle basis whose scattering off the boundary is factorizable, and so is described by

$$A_a(\theta) = A_b(-\theta)R^b_a(\theta)|B\rangle, \tag{1.2}$$

where $R^b_a(\theta)$ is the boundary scattering matrix, the boundary state takes the general form

$$|B\rangle = g\exp\left[\int_0^\infty d\theta K^{ab}(-\theta)A_a(\theta)A_b(-\theta)\right]|0\rangle, \tag{1.3}$$

where $K^{ab}(\theta) = R_{\overline{a}}^{b}(i\pi/2 - \theta)$ and $A_{\overline{a}}$ denotes the charge conjugate of A_a. However knowledge of $|B\rangle$ does not guarantee the ability to write correlation functions in a simple fashion. The fields $O_i(x_i)$ of interest may well not be simply expressible in the basis $\{A_a(\theta)\}_{a\in A}$, that is, the form factors of $O_i(x_i)$ may well be non-zero for arbitrarily high particle number.

This is the case for sine-Gordon theory. The basis of solitons $A_\pm(\theta)$ scatters factorizably off the boundary. However the fields of interest, such as the current operator or the Mandlestam fermions, in general have mode expansions involving multi-soliton states. Only at the free fermion point is this not the case. A solution to this problem may be found in choosing a different diagonalization of the Hilbert space, one that both induces factorizable scattering off the boundary and in which the fields are simply expressed. Such a basis may be found in generalizations of the spinon fields found in [6] [7] or in the anyon-super fermion fields described by [8]. However, at present it is not understood how to express sine-Gordon in terms of such fields. As such in the paper we focus on the free-fermion point where these difficulties are absent. Construction of the boundary states and computation of the correlators in this limit will set the stage for future calculations away from the free-fermion point.

At the free-fermion point, the motivating physical systems are both interacting in the bulk: the quantum Hall edges are at filling fraction $\nu = 1/2$ [1] and the quantum wires having impurity-free conductance $e^2/2h$. The free-fermion point describes interacting electrons because these physical systems are not boundary problems but impurity problems, i.e. the scattering point is in the bulk of the system. To turn the impurity problem into a boundary problem, the system is folded about the impurity. In folding the system, the interacting electrons are transformed into free ones.

The outline of this paper is as follows. In section 2 we construct the boundary states for the sine-Gordon theory on a cylinder. This construction comes in two parts. We must both compute the massless scattering matrices and the boundary entropy $g = \langle 0|B\rangle$. We do the latter in two ways, one using TBA techniques, and one via a direct calculation of a limiting form of the partition function. Using these boundary states in section 3, we compute relevant correlation functions and the partition function in full generality. This

[1] Though experiments have failed to observe edge states at $\nu = 1/2$ [9], we point out the application to quantum Hall edges to emphasize the underlying theoretical unity of quantum Hall edges with quantum wires.

partition function is then related to the partition function of a doubled boundary Ising model. In section 4 we calculate the expected ac-conductance of interacting quantum Hall edges/tunneling quantum wires. This calculation matches onto the dc-conductance calculated by [1]. In [1] the universal scaling form of the conductance was calculated by mapping the system onto a lattice model which had been solved previously by [10]. The advantage of our calculation of the conductance lies both in that it gives finite size corrections and that there is some chance it can be generalized beyond the free-fermion point.

2. Construction of Boundary States

The action for the massive sine-Gordon with an integrable boundary perturbation is

$$S_{SG} = \frac{1}{8\pi}\int_R dxdt(\partial_z\Phi\partial_{\overline{z}}\Phi + 4\lambda\cos(\widehat{\beta}\phi)) + \frac{\alpha}{4\pi}\int_B d\gamma\cos(\frac{\widehat{\beta}}{2}(\Phi-\phi_o)), \tag{2.1}$$

where $z=(t+ix)/2$, $\overline{z}=(t-ix)/2$, and B, the boundary, is described via a parametric curve, $z=\gamma(y), \overline{z}=\overline{\gamma}(y)$. This curve circulates in a positive sense around the region R over which the bulk terms are integrated. The mass term, $\lambda\cos(\widehat{\beta}\Phi)$, is included (even though we are interested in the massless limit) to mark out the basis of states we intend to employ. Only solitons scatter nicely off the boundary, i.e. factorizably, and this is the basis the mass term picks out. The other alternative, a basis organized into conformal modules labeled by primary fields, leads to a form for $|B\rangle$ vastly more complicated than (1.3). Here α is a dimensionful parameter and ϕ_o a constant. All physical quantities are independent of the sign of α. Such a sign change is implemented via $\phi_o\to\phi_o+2\pi/\widehat{\beta}$. But the subsequent shift, $\Phi\to\Phi+2\pi/\widehat{\beta}$, restores the boundary term leaving the bulk term invariant.

It is well understood that at $\widehat{\beta}=1$ the bulk portion of the sine-Gordon action is equivalent to a free Dirac fermion [11][12]. Letting $\psi_\pm$ and $\overline{\psi}_\pm$ be the left and right chiral components of the Dirac fermion with U(1) charge ± 1, the bosonisation relations are

$$\psi_\pm = \exp(\pm i\phi), \qquad \overline{\psi}_\pm = \exp(\mp i\overline{\phi}), \tag{2.2}$$

where ϕ and $\overline{\phi}$ are the chiral components of the boson Φ:

$$\begin{aligned}\phi(x,t) &= \frac{1}{2}\left(\Phi(x,t) + i\int_{-\infty}^{x} dx'\partial_t\Phi(x',t)\right);\\ \overline{\phi}(x,t) &= \frac{1}{2}\left(\Phi(x,t) - i\int_{-\infty}^{x} dx'\partial_t\Phi(x',t)\right).\end{aligned} \tag{2.3}$$

The bulk Dirac action is then given by

$$S_D^{bulk} = \frac{1}{8\pi}\int_R dxdt\,\left(\psi_+\partial_{\overline{z}}\psi_- + \psi_-\partial_{\overline{z}}\psi_+ + \overline{\psi}_-\partial_z\overline{\psi}_+ + \overline{\psi}_+\partial_z\overline{\psi}_- + 2im(\psi_-\overline{\psi}_+ - \overline{\psi}_-\psi_+)\right). \tag{2.4}$$

where a certain choice of gamma matrices[2] has been used.

In [13] the bosonisation of the boundary term was developed. This bosonization involves two terms: one implementing the boundary conditions at the free point ($\alpha = 0$), and one implementing the conditions for the interpolating perturbation ($\alpha \neq 0$). At the free point, the fermions on the boundary must satisfy

$$(\partial_y\gamma)^{1/2}\psi_\pm = e^{\pm i\sigma}(\partial_y\overline{\gamma})^{1/2}\overline{\psi}_\mp \tag{2.5}$$

where σ is a constant. The addition to the action that implements this condition is

$$S_D^{free\ bd.} = \frac{i}{8\pi}\int_B dy(e^{i\sigma}\psi_+\overline{\psi}_+ + e^{-i\sigma}\psi_-\overline{\psi}_-). \tag{2.6}$$

To construct the interpolating action, [13] assumed in the spirit of conformal perturbation theory that the fields maintain the structure that they possess at $\alpha = 0$. With this assumption, the cosine perturbation becomes

$$\begin{aligned} S_D^{int} = \alpha\int_B dy\cos(\frac{\widehat{\beta}}{2}(\Phi - \phi_o) = \alpha\int_B dy\,\Big[& e^{-i\phi_o/2}((\partial_y\gamma)^{1/2}\psi_+a_- + (\partial_y\overline{\gamma})^{1/2}\overline{\psi}_-a_+)+ \\ & e^{i\phi_0/2}((\partial_y\gamma)^{1/2}a_+\psi_- + (\partial_y\overline{\gamma})^{1/2}a_-\overline{\psi}_+)\Big], \end{aligned} \tag{2.7}$$

where $a_\pm$ are zero mode operators. They equal

$$a_\pm = \frac{1}{4}\exp(\pm i(\pi/2 + \phi - \overline{\phi})). \tag{2.8}$$

In the process of rewriting the boundary perturbation, α is renormalized. The full Dirac action with boundary is then $S_D = S_D^{bulk} + S_D^{free\ bd.} + S_D^{int.}$.

To derive the boundary state we place the boundary at $t = 0$. Then setting $\gamma(y) = iy$, $\overline{\gamma}(y) = -iy$, varying the full action with respect to the fermions and zero modes, and then eliminating the zero modes yields the following interpolating boundary conditions at $t = 0$:

$$\begin{aligned} &e^{i\phi_o}\overline{\psi}_+ - ie^{i(\phi_o-\sigma)}\psi_- - i\psi_+ + e^{-i\sigma}\overline{\psi}_- = 0 \\ &\partial_x(\overline{\psi}_- - ie^{i\sigma}\psi_+) + \alpha^2(i\overline{\psi}_- + \psi_-e^{i\phi_o}) = 0. \end{aligned} \tag{2.9}$$

[2] $\gamma^0 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $\gamma^1 = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}$

Taking $\alpha = 0$ recovers the above free boundary condition.

To derive the boundary state, we interpret the above boundary condition to vanish when acting upon it, i.e.

$$(e^{i\phi_o}\overline{\psi}_+ - ie^{i(\phi_o-\sigma)}\psi_- - i\psi_+ + e^{-i\sigma}\overline{\psi}_-)|B\rangle = 0 \text{ etc.} \tag{2.10}$$

$|B\rangle$ is expressed in terms of states from the fermionic Hilbert space. Thus we need to specify fermionic mode expansions. It is here we specialize to the massless case. Massless mode expansions for fermions on a cylinder of radius $2l$ are

$$\begin{aligned}\psi_\pm(z) &= \sqrt{\frac{1}{l}}\sum_{n=1}^{\infty}\psi_n^\pm e^{-(n-1/2)z/l} + \psi_{-n}^\pm e^{(n-1/2)z/l},\\ \overline{\psi}_\pm(\overline{z}) &= \sqrt{\frac{1}{l}}\sum_{n=1}^{\infty}\overline{\psi}_n^\pm e^{-(n-1/2)\overline{z}/l} + \overline{\psi}_{-n}^\pm e^{(n-1/2)\overline{z}/l},\end{aligned} \tag{2.11}$$

where the modes satisfy the following algebra

$$\begin{aligned}\{\psi_n^\pm, \psi_m^\pm\} &= \{\overline{\psi}_n^\pm, \overline{\psi}_m^\pm\} = \{\psi_n, \overline{\psi}_m\} = 0\\ \{\psi_n^-, \psi_m^+\} &= \{\overline{\psi}_n^-, \overline{\psi}_m^+\} = \delta_{n+m,0}.\end{aligned} \tag{2.12}$$

The normalization of the mode expansions is fixed by insisting as $z \to 0$

$$\begin{aligned}\langle\psi_-(z)\psi_+(0)\rangle &= \frac{1}{z} + \cdots;\\ \langle\overline{\psi}_-(\overline{z})\overline{\psi}_+(0)\rangle &= \frac{1}{\overline{z}} + \cdots.\end{aligned} \tag{2.13}$$

The above mode expansions are anti-periodic, i.e. we are in the Neveu-Schwarz sector. A priori we would expect a contribution to the boundary state $|B\rangle$ from the Ramond sector. This is the case with the Ising model (see [14] [15]). However as will be shown, the boundary entropy for the Ramond sector is zero in the massless limit, and so this sector makes no contribution to $|B\rangle$.

The boundary state takes the general form

$$|B\rangle = g(\alpha,l)\exp\left[\sum_{n=1}^{\infty} a_n\overline{\psi}_{-n}^-\psi_{-n}^- + b_n\overline{\psi}_{-n}^+\psi_{-n}^+ + c_n\overline{\psi}_{-n}^+\psi_{-n}^- + d_n\overline{\psi}_{-n}^-\psi_{-n}^+\right]|0\rangle, \tag{2.14}$$

where $g(\alpha,l)$ is the boundary entropy. In substituting the mode expansions into (2.10), expanding out $|B\rangle$ to first order, and solving, we find,

$$\begin{aligned}a_n &= ie^{-i\sigma}/(1+\lambda_n) \qquad b_n = ie^{i\sigma}/(1+\lambda_n)\\ c_n &= i\lambda_n e^{i\phi_o}/(1+\lambda_n) \qquad d_n = i\lambda_n e^{-i\phi_o}/(1+\lambda_n),\end{aligned} \tag{2.15}$$

where $\lambda_n = \alpha^2 l/(n-1/2)$. It now remains to compute $g(\alpha, l)$.

The boundary entropy is easily derived from knowledge of the partition function. Consider the partition function for the theory on a cylinder of radius $2l$ and length R. (These dimensions for the cylinder will remain the same throughout the paper.) Interpreting the length R to be in the time direction, the partition function in the limit $R \gg 2l$ is given simply in terms of the g-factors:

$$Z = g_a(\alpha, l) g_b(\alpha, l) e^{-RE_o}, \tag{2.16}$$

where a and b denote the boundary conditions on the two ends of the cylinder and E_o is the ground state energy of the system. The calculation of the partition function in this limit will be done in two ways. The first calculates the partition function directly while the second uses TBA techniques to access it.

To find the partition function, we keep the same limit $R \gg 2l$ but reinterpret the axis of the cylinder as space. The partition function then equals

$$Z_{\pm} = \mathrm{Tr}(\pm 1)^F e^{-4\pi l H} = e^{-4\pi l E_o} \prod_k (1 \pm e^{-4\pi l k}), \tag{2.17}$$

where $\pm$ indicate anti-periodic (Neveu-Schwarz)/periodic (Ramond) boundary conditions, $E_o = -1/2 \sum_k 4\pi l k$, and the product $\prod_k$ is over all allowed modes. To determine the allowed modes we apply the boundary conditions.

In this case there are two sets of boundary conditions: one at $t=0$ and one at $t=R$:

$$\begin{aligned}
0 &= \overline{\psi}_+ - i\psi_- - i\psi_+ + \overline{\psi}_-|_{t=0};\\
0 &= \partial_x(\overline{\psi}_- - i\psi_+) + \alpha^2(i\overline{\psi}_- + \psi_-)|_{t=0};\\
0 &= \overline{\psi}_+ + i\psi_- + i\psi_+ + \overline{\psi}_-|_{t=R};\\
0 &= \partial_x(\overline{\psi}_- + i\psi_+) + \alpha^2(i\overline{\psi}_- - \psi_-)|_{t=R}.
\end{aligned} \tag{2.18}$$

The last two conditions arise because the boundary parametrization at $t=R$ is $\gamma(y) = -iy$. In these expressions both σ and ϕ_o have been set to zero. σ can always be gauged away [5][13], and $\phi_o \to \phi_o + c$ is a symmetry in the massless limit.

As it stands the fields $\psi_{\pm}$,$\overline{\psi}_{\pm}$ are not independent. A change of basis facilitates the determination of their interdependence. We write

$$T_{\pm} = \psi_+ \pm \psi_-, \quad \overline{T}_{\pm} = \overline{\psi}_+ \pm \overline{\psi}_-. \tag{2.19}$$

The boundary conditions then become

$$
\begin{aligned}
0 &= \overline{T}_+ - iT_+|_{t=0}; \\
0 &= \partial_x(\overline{T}_- + iT_-) + \alpha^2(i\overline{T}_- + T_-)|_{t=0}; \\
0 &= \overline{T}_+ + iT_+|_{t=R}; \\
0 &= \partial_x(\overline{T}_- - iT_-) + \alpha^2(i\overline{T}_- - T_-)|_{t=R}.
\end{aligned}
\tag{2.20}
$$

The boundary conditions clearly separate with this change of basis.

In the mode expansions

$$
\begin{aligned}
T^{\pm} &= \sum_{k_\pm} t^{\pm}_{k_\pm} e^{ik_\pm(t+ix)}, \\
\overline{T}^{\pm} &= \sum_{k_\pm} \overline{t}^{\pm}_{k_\pm} e^{-ik_\pm(t-ix)},
\end{aligned}
\tag{2.21}
$$

the above boundary conditions constrain the allowed values of $k_\pm$. Substituting the mode expansions in, we find the following :

$$
\begin{aligned}
0 &= 1 + X_\pm(k_\pm); \\
X_+ &= e^{-2iRk}; \\
X_- &= e^{-2iRk}\frac{(\alpha^2 - ik)^2}{(\alpha^2 + ik)^2}.
\end{aligned}
\tag{2.22}
$$

We see then that the t_+ modes are free while the t_- modes are interacting.

The partition function is then given by

$$
\log Z_\pm = \sum_{k_+>0} \left[2\pi l k_+ + \log(1 \pm e^{-4\pi l k_+})\right] + (k_+ \to k_-). \tag{2.23}
$$

These sums can be evaluated using Matsubara sum techniques developed in [14]. It is then straightforward to extract expressions for the boundary entropy. We relegate the details to an appendix. The result is

$$
\begin{aligned}
\log g_\pm(\alpha, l) &= \frac{1}{\pi}\int_0^\infty \frac{1}{1+k^2}\left[\log(1 \pm e^{-2\pi a k}) + \lim_{b\to 0}\log(1 \pm e^{-2\pi b k})\right] + \frac{1 \mp 1}{4}\log(2) \\
&= \begin{cases} \log\left[2\sqrt{\pi}a^a/(\Gamma(a+1/2)e^a)\right] & + \\ -\infty & -, \end{cases}
\end{aligned}
\tag{2.24}
$$

where again $a = 2l\alpha^2$. Thus in the Ramond sector, the boundary entropy is zero. These results can be compared with calculations of the boundary entropy, g^I, of a boundary Ising model in a magnetic field of strength α. In [15][14] g^I was found to be

$$\begin{aligned} g^I_+(\alpha) &= \frac{\sqrt{2\pi}}{\Gamma(a+1/2)}\left(\frac{a}{e}\right)^a; \\ g^I_-(\alpha) &= 2^{1/4}\frac{\sqrt{2\pi a}}{\Gamma(a+1)}\left(\frac{a}{e}\right)^a. \end{aligned} \tag{2.25}$$

Thus we have

$$g_\pm(\alpha) = g^I_\pm(\alpha) g^I_\pm(0). \tag{2.26}$$

Hence we have verified at the level of boundary entropy the results indicated in [10]: boundary sine-Gordon is equivalent to two copies of a boundary Ising model, one copy in a boundary field of strength α and one with zero field. We will demonstrate this equivalence at the level of partition functions in section 3.

We now go on to calculate the boundary entropy using TBA techniques. To implement the TBA analysis, we need the boundary and bulk scattering matrices to be diagonal. This is not the case for the sine-Gordon theory, but it is easily rectified through a change of basis. In the soliton/anti-soliton basis, the boundary scattering is described by

$$\begin{aligned} A^\dagger_+(\theta)\mathrm{B} &= P^+(\theta)A^\dagger_+(-\theta) + Q^+(\theta)A^\dagger_-(-\theta)\mathrm{B}; \\ A^\dagger_-(\theta)\mathrm{B} &= P^-(\theta)A^\dagger_-(-\theta) + Q^-(\theta)A^\dagger_+(-\theta)\mathrm{B}; \end{aligned} \tag{2.27}$$

where $A^\dagger_\pm$ are Faddeev-Zamolodchikov operators which create solitons/anti-solitons. In [13] the boundary scattering matrices were found to be:

$$\begin{aligned} P^\pm(\theta) \equiv P(\theta) &= \left((1-\gamma/2)\cosh(\theta)\right)/D(\theta); \\ Q^\pm(\theta) \equiv Q(\theta) &= -i\sinh(2\theta)/2D(\theta); \end{aligned} \tag{2.28}$$

where

$$\begin{aligned} \gamma &= 2\alpha^2/m; \\ D(\theta) &= i\gamma\cosh(\frac{\theta+i\pi/2}{2})\sinh(\frac{\theta-i\pi/2}{2}) - \cosh^2(\theta), \end{aligned} \tag{2.29}$$

and again both σ and ϕ_o have been set to zero. Although we are interested in the massless limit, it is easier to work in the massive case, taking $m \to 0$ only at the end.

To diagonalize the scattering off the boundary, we introduce the operators in analogy with (2.19):

$$T^\dagger_\pm = A^\dagger_+ \pm A^\dagger_-. \tag{2.30}$$

As is easily seen, this basis scatters diagonally off the boundary:

$$T_\pm^\dagger(\theta)|B\rangle = \left(P(\theta) \pm Q(\theta)\right) T_\pm^\dagger(-\theta)|B\rangle \equiv R_\pm(\theta) T_\pm^\dagger(-\theta)|B\rangle. \tag{2.31}$$

It is also easily seen that scattering in this new basis is unitary, i.e. $R_\pm(\theta)R_\pm(-\theta) = 1$, important as the TBA analysis requires it.

We again keep the same limit $R \gg 2l$ and the interpretation of the axis of the cylinder as space. Following [16], the system can be described as a widely spaced set of n particles of definite rapidities, θ_i, types c_i, and located in regions x_i. Because the particles are widely spaced, the particles move as free ones and we can ignore off-mass shell effects. Hence we can describe the system via a wavefunction $\Psi(\theta_i, c_i, x_i)$. Knowledge of the scattering matrices allows to constrain the wavefunction arrived at through the interchange of two adjacent particles:

$$\begin{aligned} &\Psi(\cdots;\theta_i, c_i, x_i; \theta_{i+1}, c_{i+1}, x_{i+1};\cdots) \\ &\qquad\qquad = S_{i,i+1}(\theta_i - \theta_{i+1})\Psi(\cdots;\theta_{i+1}, c_{i+1}, x_i; \theta_i, c_i, x_{i+1};\cdots) \end{aligned} \tag{2.32}$$

or through the scattering of a particle off the two boundaries a and b:

$$\begin{aligned} \Psi(x_1, \theta_1, c_1, \cdots) &= R_1^a(\theta_1)\Psi(x_1, -\theta_1, c_1, \cdots); \\ \Psi(\cdots, x_n, \theta_n, c_n) &= (R_n^b)^{-1}(\theta_n)\Psi(\cdots, x_n, -\theta_n, c_n), \end{aligned} \tag{2.33}$$

where S_{ij} describes scattering of particles of types c_i with c_j (only two indices are needed as the scattering is diagonal) and R_i describes scattering of particles of type c_i off the boundary. Scattering the i-th particle of rapidity $\theta_i \neq 0$ up the cylinder, back down, and then up to its original location leads to the quantization condition:

$$e^{-2im_i R\sinh\theta_i}\prod_{i\neq j} S_{ij}(\theta_i - \theta_j)S_{ij}(\theta_i + \theta_j)R_i^a(\theta_i)R_i^b(\theta_i) = 1. \tag{2.34}$$

Writing $\rho_i(\theta)$ as the density of levels for particles of type c_i, and $\widetilde{\rho}_i(\theta)$ as the density of occupied states for particles of type i, this quantization condition can be recast as

$$\begin{aligned} 2\pi\rho_i(\theta) = mR\cosh(\theta) &+ i\sum_j \int_{-\infty}^{\infty} d\theta' \widetilde{\rho}(\theta')\partial_\theta \log S_{ij}(\theta - \theta') \\ &+ \frac{i}{2}\partial_\theta \log(R_i^a(\theta)R_i^b(\theta)) - \frac{i}{2}\partial_\theta \log S_{ii}(2\theta) + \pi\delta(\theta). \end{aligned} \tag{2.35}$$

$\theta = 0$ is not an allowed solution to the quantization condition. Thus $\delta(\theta)$ is included (by hand) to remove this contribution.

If the system is bosonic, its entropy is given by

$$S_b = \sum_i \int_{-\infty}^{\infty} d\theta \left[(\widetilde{\rho}_i + \rho_i) \log(\rho_i + \widetilde{\rho}_i) - \rho_i \log \rho_i - \widetilde{\rho}_i \log \widetilde{\rho}_i \right], \tag{2.36}$$

while if it is fermionic, the entropy is

$$S_f = \sum_i \int_{-\infty}^{\infty} d\theta \left[(\widetilde{\rho}_i - \rho_i) \log(\rho_i - \widetilde{\rho}_i) + \rho_i \log \rho_i - \widetilde{\rho}_i \log \widetilde{\rho}_i \right]. \tag{2.37}$$

In either case the system's energy is

$$H = \sum_i \int_{-\infty}^{\infty} d\theta m_i R \cosh(\theta) \widetilde{\rho}_i. \tag{2.38}$$

Finding the extremum of the free energy

$$\log Z_{b/f} = -4\pi l F_{b/f} = -4\pi l H + S_{b/f}, \tag{2.39}$$

by varying $\widetilde{\rho}_i$ and ρ_i and using the quantization condition leads to the following set of integral equations:

$$\begin{aligned} \epsilon_i^{b/f}(\theta) &= 4\pi l m_i \cosh\theta \mp \frac{1}{2\pi i} \sum_j \int_{-\infty}^{\infty} d\theta' \partial_{\theta'} \log S_{ij}(\theta' - \theta) \log(1 \mp e^{-\epsilon_j^{b/f}(\theta')}); \\ \log Z_{b/f} &= \mp \sum_i \int_{-\infty}^{\infty} \frac{d\theta}{2\pi} \log(1 \mp e^{-\epsilon_i^{b/f}(\theta)})(mR\cosh\theta + \\ &\qquad \frac{i}{2}\partial_\theta \log(R_i^a(\theta) R_i^b(\theta)) - \frac{i}{2}\partial_\theta \log S_{ij}(2\theta) + \pi\delta(\theta)). \end{aligned} \tag{2.40}$$

where we have introduced pseudo-energies, $\epsilon_i^{b/f}$, given by

$$\epsilon_i^{b/f} = \frac{\widetilde{\rho}_i}{\rho_i \pm \widetilde{\rho}_i}. \tag{2.41}$$

The pieces in the equation for $\log Z_{b/f}$ corresponding to the boundary entropy are those that do not scale with R. Thus

$$\begin{aligned} \log g_a^{b/f} + \log g_b^{b/f} &= \mp \sum_i \int_{-\infty}^{\infty} \frac{d\theta}{4\pi} \log(1 \mp e^{-\epsilon_i^{b/f}(\theta)}) \times \\ &\quad (i\partial_\theta \log(R_i^a(\theta) R_i^b(\theta)) - i\partial_\theta \log S_{ii}(2\theta) + 2\pi\delta(\theta)). \end{aligned} \tag{2.42}$$

In the Neveu-Schwarz sector of the theory, the fermions fill the levels as fermions. Thus $\log g_{NS} = \log g_f$. Examining the partition function in (2.17) for the Ramond sector,

we see it is equivalent to the inverse of a bosonic partition function. So $\log g_R = -\log g_b$. As the theory is trivial in the bulk (i.e. $S = -1$), $\epsilon_i^{b/f}(\theta) = 4\pi l m \cosh\theta$. So the boundary entropy for the two sectors is given by

$$\log g_\pm \equiv \log g_{NS/R} = \int_{-\infty}^{\infty} d\theta \log(1 \pm e^{-4\pi l m \cosh\theta}) \times \left(\frac{i}{4\pi}\partial_\theta \log(P^2(\theta) - Q^2(\theta)) + \frac{1}{2}\delta(\theta)\right). \tag{2.43}$$

This expression differs by an overall sign from that used in [15] to compute the boundary entropy of the boundary Ising model. This change in sign results from a difference in sign conventions used for the mass in deriving the scattering matrices. Doing the integrals and taking the massless limit leads to

$$\begin{aligned} \log g_+ &= \log\left[2\sqrt{\pi}a^a/\Gamma(a+1/2)e^a\right]; \\ \log g_- &= -\infty, \end{aligned} \tag{2.44}$$

where $a = 2l\alpha^2$. We thus see that the TBA analysis reproduces exactly the results of the direct calculation. In general, TBA only can reproduce the boundary entropy up to a constant. Corrections arise both from the use of Stirling's formula in the expressions for the entropy and from off-mass shell effects. But apparently, TBA is exact in this case.

3. Correlation Functions and Partition Functions

3.1. Correlation Functions

Given below are the two-point functions together with the current-current correlators. First we give the two-point functions unaffected by the boundary, the left-left and right-right fermionic correlators:

$$\begin{aligned} \langle\overline{\psi}^\pm(x,\tau)\overline{\psi}^\mp(x',\tau')\rangle &= \frac{1}{l}\left[\theta(\tau-\tau')\frac{e^{-\bar{s}/4l}}{1-e^{-\bar{s}/2l}} - \theta(\tau'-\tau)\frac{e^{\bar{s}/4l}}{1-e^{\bar{s}/2l}}\right]; \\ \langle\psi^\pm(x,\tau)\psi^\mp(x',\tau')\rangle &= \frac{1}{2l}\left[\theta(\tau-\tau')\frac{e^{-s/4l}}{1-e^{-s/2l}} - \theta(\tau'-\tau)\frac{e^{s/4l}}{1-e^{s/2l}}\right], \end{aligned} \tag{3.1}$$

where $s = \tau - \tau' + i(x - x')$. The right-left left-right fermionic two point functions, on the other hand, are affected by the boundary. No longer zero, they couple to the two particle

contribution $|B\rangle$, and are given by:

$$\begin{aligned}
\langle\overline{\psi}^{\pm}(x,\tau)\psi^{\pm}(x',\tau')\rangle &= -\frac{1}{l}\sum_{n\in Z^+}\frac{ie^{\mp i\sigma}}{1+\lambda_n}e^{-(n-1/2)\overline{y}/2l};\\
\langle\psi^{\pm}(x,\tau)\overline{\psi}^{\pm}(x',\tau')\rangle &= \frac{1}{l}\sum_{n\in Z^+}\frac{ie^{\mp i\sigma}}{1+\lambda_n}e^{-(n-1/2)y/2l};\\
\langle\overline{\psi}^{\mp}(x,\tau)\psi^{\pm}(x',\tau')\rangle &= -\frac{1}{l}\sum_{n\in Z^+}\frac{ie^{\pm i\phi_o}\lambda_n}{1+\lambda_n}e^{-(n-1/2)\overline{y}/2l};\\
\langle\psi^{\pm}(x,\tau)\overline{\psi}^{\mp}(x',\tau')\rangle &= \frac{1}{l}\sum_{n\in Z^+}\frac{ie^{\pm i\phi_o}\lambda_n}{1+\lambda_n}e^{-(n-1/2)y/2l}.
\end{aligned}\tag{3.2}$$

where $y=\tau+\tau'+i(x-x')$. The current-current correlators are then given in terms of these two-point functions:

$$\begin{aligned}
\langle j_r(x,\tau)j_r(x',\tau')\rangle &= \frac{1}{(4\pi)^2}\langle\overline{\psi}^-(x,\tau)\overline{\psi}^+(x',\tau')\rangle\langle\overline{\psi}^+(x,\tau)\overline{\psi}^-(x',\tau')\rangle\\
&= \frac{1}{(4l\pi)^2}\left[\theta(\tau-\tau')\frac{e^{-\overline{s}/2l}}{(1-e^{-\overline{s}/2l})^2}+\theta(\tau'-\tau)\frac{e^{\overline{s}/2l}}{(1-e^{\overline{s}/2l})^2}\right];\\
\langle j_r(x,\tau)j_r(x',\tau')\rangle &= \frac{1}{(4\pi)^2}\langle\overline{\psi}^-(x,\tau)\overline{\psi}^+(x',\tau')\rangle\langle\overline{\psi}^+(x,\tau)\overline{\psi}^-(x',\tau')\rangle\\
&= \frac{1}{(4l\pi)^2}\left[\theta(\tau-\tau')\frac{e^{-s/2l}}{(1-e^{-s/2l})^2}+\theta(\tau'-\tau)\frac{e^{s/2l}}{(1-e^{s/2l})^2}\right];
\end{aligned}\tag{3.3}$$

$$\begin{aligned}
\langle j_r(x,\tau)j_l(x',\tau')\rangle &= \frac{1}{(4\pi)^2}\Big(\langle\overline{\psi}^+(x,\tau)\psi^-(x',\tau')\rangle\langle\overline{\psi}^-(x,\tau)\psi^+(x',\tau')\rangle\\
&\qquad -\langle\overline{\psi}^+(x,\tau)\psi^+(x',\tau')\rangle\langle\overline{\psi}^-(x,\tau)\psi^-(x',\tau')\rangle\Big)\\
&= \frac{1}{(4l\pi)^2}\sum_{k,k'\in Z^+}\frac{1-\lambda_k\lambda_{k'}}{(1+\lambda_k)(1+\lambda'_k)}e^{-(k+k'-1)\overline{y}/2l};\\
\langle j_l(x,\tau)j_r(x',\tau')\rangle &= \frac{1}{(4\pi)^2}\Big(\langle\psi^-(x,\tau)\overline{\psi}^+(x',\tau')\rangle\langle\psi^+(x,\tau)\overline{\psi}^-(x',\tau')\rangle\\
&\qquad -\langle\psi^-(x,\tau)\overline{\psi}^-(x',\tau')\rangle\langle\psi^+(x,\tau)\overline{\psi}^+(x',\tau')\rangle\Big)\\
&= \frac{1}{(4l\pi)^2}\sum_{k,k'\in Z^+}\frac{1-\lambda_k\lambda_{k'}}{(1+\lambda_k)(1+\lambda'_k)}e^{-(k+k'-1)y/2l};
\end{aligned}\tag{3.4}$$

3.2. Calculation of Partition Functions

We have already demonstrated that at the level of boundary entropy, the boundary sine-Gordon is equivalent to two copies of boundary Ising, one copy with a free boundary

and one copy with a boundary in a magnetic field of strength α. We now show this equivalence holds at the level of partition functions.

With time along the axis of the cylinder, the partition function in the presence of two boundaries is given by

$$Z_{\alpha'\alpha} = \langle \mathrm{B}(\alpha')|e^{-HR}|\mathrm{B}(\alpha)\rangle, \tag{3.5}$$

where H is the Hamiltonian on the cylinder:

$$\begin{aligned} H &= \frac{1}{2l}(L_0 + \overline{L}_0 - \frac{c}{12}) \\ &= \sum_{n\in Z^+} \frac{n-1/2}{2l}\left(\psi^-_{-n}\psi^-_n + \psi^+_{-n}\psi^+_n + \overline{\psi}^-_{-n}\overline{\psi}^-_n + \overline{\psi}^+_{-n}\overline{\psi}^+_n\right) - \frac{1}{24l} \end{aligned} \tag{3.6}$$

.

To compute the inner product in (3.4), we use the formula

$$\langle 0|e^{(\widetilde{a},Ma)}e^{(a^\dagger,N\widetilde{a}^\dagger)}|0\rangle = \det(1+NM), \tag{3.7}$$

where $(\widetilde{a}, Ma) = \sum_{nm} \widetilde{a}_n M_{nm} a_m$, $\{a_n, a^\dagger_m\} = \{\widetilde{a}_n, \widetilde{a}^\dagger_m\} = \delta_{n+m,0}$, and $\{\widetilde{a}_n, a^\dagger_m\} = 0$. We thus obtain

$$Z_{\alpha'\alpha} = g_+(\alpha)g_+(\alpha')q^{-1/24}\prod_{n=1}^{\infty}(1+q^{n-1/2})(1+\frac{(1-\lambda_n)(1-\lambda'_n)}{(1+\lambda_n)(1+\lambda'_n)}q^{n-1/2}), \tag{3.8}$$

where $q = e^{-R/l}$ and $\lambda_n = \alpha^2 l/(n-1/2)$.

The partition function for an Ising model with magnetic fields α, α' on the boundaries was computed in [15]:

$$\begin{aligned} Z^I_{\alpha\alpha'} =& \frac{1}{2}q^{-1/48}g^I_+(\alpha)g^I_+(\alpha')\prod_{n=1}^{\infty}\left(1 + a_{n-1/2}(\alpha)a_{n-1/2}(\alpha')q^{n-1/2}\right) \\ &+ \frac{1}{2}\mathrm{sgn}(\alpha\alpha')q^{1/24}g^I_-(\alpha)g^I_-(\alpha')\prod_{n=1}^{\infty}\left(1 + a_{n-1}(\alpha)a_{n-1}(\alpha')q^{n-1}\right), \end{aligned} \tag{3.9}$$

where

$$\begin{aligned} a_n(\alpha) &= \frac{1-\alpha^2 l/n}{1+\alpha^2 l/n}, \\ a &= 2l\alpha^2, \end{aligned} \tag{3.10}$$

and $g^I_\pm$ are given in (2.23). The first term in $Z^I_{\alpha\alpha'}$ is the contribution from the Neveu-Schwarz sector of the theory and the second term is the contribution from the Ramond

sector (zero in the case of boundary sine-Gordon). We can then make the following identification:

$$Z_{\alpha'\alpha} = 2Z^I_{00}\left(Z^I_{\alpha\alpha'} + Z^I_{\alpha,-\alpha'}\right), \tag{3.11}$$

where we have used the fact $g_+(\alpha) = g^I_+(\alpha)g^I_+(\alpha')$. Summing $Z^I_{\alpha\alpha'}$ with $Z^I_{\alpha,-\alpha'}$ mods out the Ramond sector of the Ising theory. This cancellation reflects the indifference of boundary sine-Gordon to the sign of α.

At the conformal boundary points (i.e. $\alpha = 0$ and $\alpha = \infty$) the partition function of the boundary sine-Gordon can be expressed in terms of the $c = 1/2$ characters. The corresponding expressions for boundary Ising are well known [17]:

$$Z^I_{ab} = \sum_{ij} n^i_{ab} S^j_i X_j(q), \tag{3.12}$$

where a and b label the boundary conditions, S governs modular transformation of the $c = 1/2$ characters X_j, $j = 0, 1/2, 1/16$, and n^i_{ab} is the number of times that the irreducible representation of highest weight i appears in the spectrum of the cross-channel Hamiltonian with boundary conditions a and b. S is given by

$$S = \frac{1}{2}\begin{pmatrix} 1 & 1 & -\sqrt{2} \\ 1 & 1 & \sqrt{2} \\ \sqrt{2} & -\sqrt{2} & 0 \end{pmatrix}, \tag{3.13}$$

and the non-zero values of n^i_{ab} (all equal to 1) are $n^0_{\pm\infty,\pm\infty}$, n^0_{00}, $n^{1/2}_{00}$, $n^{1/2}_{\pm\infty,\mp\infty}$, and $n^{1/16}_{\pm\infty,0}$. The $c = 1/2$ characters are given explicitly by

$$\begin{aligned} \chi_0(q) &= \frac{q^{-1/48}}{2}\left[\prod_{n=1}^{\infty}(1+q^{n-1/2}) + \prod_{n=1}^{\infty}(1-q^{n-1/2})\right]; \\ \chi_{1/2}(q) &= \frac{q^{-1/48}}{2}\left[\prod_{n=1}^{\infty}(1+q^{n-1/2}) - \prod_{n=1}^{\infty}(1-q^{n-1/2})\right]; \\ \chi_{1/16}(q) &= q^{1/2}\prod_{n=0}^{\infty}(1+q^n). \end{aligned} \tag{3.14}$$

Then using (3.11) and (3.12), the conformal points of the partition function for boundary sine-Gordon equal:

$$\begin{aligned} Z_{00} &= 4\left(\chi_0(q) + \chi_{1/2}(q)\right)^2; \\ Z_{0,\pm\infty} &= 2^{3/2}\left(\chi_0^2(q) - \chi_{1/2}^2(q)\right); \\ Z_{\pm\infty,\pm\infty} &= Z_{\pm\infty,\mp\infty} = 2\left(\chi_0(q) + \chi_{1/2}(q)\right)^2. \end{aligned} \tag{3.15}$$

These formulas can be directly verified by taking the appropriate limits in (3.8).

4. Calculation of AC-Conductance

Using the results of the previous sections, we now go on to calculate the conductance of a tunneling quantum wire with free conductance $e^2/2h$, or equivalently, two interacting quantum Hall edges at $\nu = 1/2$. We begin by mapping these systems onto the boundary sine-Gordon at $\widehat{\beta} = 1$. The Hamiltonian for an impurity free wire/two non-interacting edges is

$$H_0 = -\frac{v}{4\pi\nu} \int_{-R}^{R} dx (\partial_x \phi_L)^2 + (\partial_x \phi_R)^2, \tag{4.1}$$

where ϕ_L and ϕ_R are left and right moving chiral bosons. The system has length $2R$ and its excitations have velocity v. Henceforth we set $v = 1$.

We now allow the right and left movers to interact. This interaction is realized through an impurity at the origin:

$$H_{imp} = -\left[\frac{\alpha}{2} e^{i\phi_L - i\phi_R} + \frac{\alpha}{2} e^{i\phi_R - i\phi_L}\right]_{x=0}. \tag{4.2}$$

This impurity scatters left movers of charge νe into right movers of the same charge. For the value of ν we are interested in, it is the only relevant operator which can induce scattering [1].

Mapping $H_o + H_{imp}$ onto boundary sine-Gordon is done in two steps [4][3]. First a change of basis is made and a spurious degree of freedom is removed. Secondly, the system is folded, changing it from an impurity problem to a boundary problem. As H_{imp} depends only on the combination $\phi_L - \phi_R$, the following change of basis is suggested:

$$\phi^{e/o} = (\phi_L(x) \pm \phi_R(-x)). \tag{4.3}$$

Both ϕ_e and ϕ_o are left movers. In this basis $H = H_0 + H_{imp}$ becomes

$$H = -\frac{1}{16\pi\nu} \int_{-R}^{R} dx (\partial_x \phi_e)^2 + (\partial_x \phi_o)^2 - \left[\frac{\alpha}{2} e^{i\phi_o(x=0)} + \frac{\alpha}{2} e^{-i\phi_o(x=0)}\right] \tag{4.4}$$

We see that while ϕ_o is interacting ϕ_e is not. ϕ_e can thus be dropped with H reducing to:

$$H = -\frac{1}{16\pi\nu} \int_{-R}^{R} dx (\partial_x \phi_o)^2 - \left[\frac{\alpha}{2} e^{i\phi_o(x=0)} + \frac{\alpha}{2} e^{-i\phi_o(x=0)}\right] \tag{4.5}$$

This is the first step of the mapping.

To implement the second step, folding the system, we separate out the degrees of freedom of ϕ_o defined on $-R < x < 0$ from ϕ_o defined on $0 < x < R$ through the variables:

$$\begin{cases} \phi_l(x) = \phi_o(x) & 0 < x < R \\ \phi_r(-x) = -\phi_o(x) & -R < x < 0 \ . \end{cases} \tag{4.6}$$

ϕ_l (ϕ_r) is a left (right) chiral boson. The minus sign present in the definition of ϕ_r transforms the boson in the unfolded theory, $\phi_L - \phi_R$, into its dual folded counterpart, $\phi_l + \phi_r$. Under the change of variables, H becomes

$$H = -\frac{1}{16\pi\nu}\int_0^R dx(\partial_x\phi_r)^2 + (\partial_x\phi_l)^2 - \frac{\alpha}{2}\left[e^{i\phi_l(x=0)} + e^{-i\phi_l(x=0)}\right] \tag{4.7}$$

Because of the boundary, ϕ_l and ϕ_r are not independent. In the limit $\alpha = 0$ we have [13]

$$\phi_l(x=0) = \phi_r(x=0) - \sigma. \tag{4.8}$$

Setting σ to zero and treating the boundary term in conformal perturbation theory then allows us to write

$$H = \frac{1}{32\pi\nu}\int_0^R dx(\partial_z\Phi)^2 + (\partial_{\overline{z}}\Phi)^2 - \alpha\cos(\frac{\Phi}{2})|_{x=0}, \tag{4.9}$$

where $\Phi = \phi_l + \phi_r$. For $\nu = 1/2$ this is precisely the Hamiltonian of boundary sine-Gordon at the free-fermion point.

To compute the ac-conductance of the system we use a Kubo type formula:

$$G(w) = \frac{1}{(2R)^2}\frac{ie^2}{\hbar w}\int_{-R}^R dxdx'\int_0^\beta d\tau e^{iw_m\tau}\langle j_1(x,\tau)j_1(x',0)\rangle|_{w_m=-iw+\epsilon} \tag{4.10}$$

where $\beta = 4\pi l$ is the inverse temperature and $\langle j_1(x,\tau)j_1(x',0)\rangle$ is a temperature Green function of the spatial current in the unfolded system. As such we need to relate it to the correlators in the folded system. j_1 is given by

$$j_1 = -\frac{i}{2\pi}\partial_t\left(\phi_L - \phi_R\right). \tag{4.11}$$

Hence in terms of the currents in the folded system, $j_l = -\frac{i}{2\pi}\partial_t\phi_l$, $j_r = -\frac{i}{2\pi}\partial_t\phi_r$, we have

$$\begin{aligned}\int_{-R}^R dxdx'\langle j_1(x,\tau)j_1(x',0)\rangle = \int_0^R dxdx'\langle j_l(x,\tau)j_l(x',0)\rangle + \langle j_r(x,\tau)j_r(x',0)\rangle \\ \langle j_l(x,\tau)j_r(x',0)\rangle + \langle j_r(x,\tau)j_l(x',0)\rangle\end{aligned} \tag{4.12}$$

The correlators on the r.h.s. of (4.12) are obtained from those in section 3 by interchanging time and space (in this case the boundary is in space). With $x, x' > 0$, we obtain

$$\begin{aligned}
\langle j_r(x,\tau) j_r(x',0)\rangle &= \frac{1}{(4l\pi)^2}\left[\theta(x-x')\frac{e^{-\overline{w}/2l}}{(1-e^{-\overline{w}/2l})^2} + \theta(x'-x)\frac{e^{\overline{w}/2l}}{(1-e^{\overline{w}/2l})^2}\right];\\
\langle j_l(x,\tau) j_l(x',\tau')\rangle &= \frac{1}{(4l\pi)^2}\left[\theta(x-x')\frac{e^{-w/2l}}{(1-e^{-w/2l})^2} + \theta(x'-x)\frac{e^{w/2l}}{(1-e^{w/2l})^2}\right];\\
\langle j_r(x,\tau) j_l(x',\tau')\rangle &= \frac{1}{(4l\pi)^2}\sum_{k,k'\in Z^+}\frac{1-\lambda_k\lambda_{k'}}{(1+\lambda_k)(1+\lambda_k')}e^{-(k+k'-1)\overline{u}/2l};\\
\langle j_l(x,\tau) j_r(x',\tau')\rangle &= \frac{1}{(4l\pi)^2}\sum_{k,k'\in Z^+}\frac{1-\lambda_k\lambda_{k'}}{(1+\lambda_k)(1+\lambda_k')}e^{-(k+k'-1)u/2l};
\end{aligned} \tag{4.13}$$

where $w = x - x' + i\tau$ and $u = x + x' + i\tau$.

By taking the Matsubara decomposition of these correlators and then analytically continuing, $w_n \to -iw + \epsilon$, we obtain

$$\begin{aligned}
\langle j_r(x) j_r(x')\rangle(w) &= -\frac{iw}{2\pi}\theta(x'-x)e^{iw(x'-x)};\\
\langle j_l(x) j_l(x')\rangle(w) &= -\frac{iw}{2\pi}\theta(x-x')e^{iw(x-x')};\\
\langle j_r(x) j_l(x')\rangle(w) &= 0;\\
\langle j_l(x) j_r(x')\rangle(w) &= \frac{i}{4\pi l}\int_{-\infty}^{\infty} dk(1-f(k))(1-f(2lw-k))\times\\
&\qquad \frac{k(k-2lw)-\delta^2}{(i(k-2lw)+\delta)(\delta-ik)}(e^{-4\pi wl}-1)e^{iw(x+x')};
\end{aligned} \tag{4.14}$$

where $f(k) = (1+e^{2\pi k})^{-1}$ and $\delta = \alpha^2 l$. This last two expressions are derived in Appendix B. $\langle j_r j_l\rangle$ only has negative Matsubara frequencies. Hence in the analytic continuation to the retarded Green function, it vanishes.

Putting everything together we find for $G(w)$:

$$\begin{aligned}
\mathrm{Re}G(w) &= \frac{e^2}{h}\frac{\sin^2(wR/2)}{R^2w^2}\left[1+\frac{e^{4\pi lw}-1}{2lw}\times\right.\\
&\left.\int_{-\infty}^{\infty} dk\frac{\left((k^2-2lwk)^2-\delta^4\right)\cos(wR)-2lw\delta\sin(wR)}{(k^2+\delta^2)((k-2lw)^2+\delta^2)}f(k)f(2lw-k)\right];\\
\mathrm{Im}G(w) &= \frac{e^2}{2h}\frac{1}{R^2w^2}\left[Rw-\sin(wR)+(e^{4\pi lw}-1)\sin^2(wR)\times\right.\\
&\left.\int_{-\infty}^{\infty} dk\frac{2lw\delta\cos(wR)+((k^2-2lwk)^2-\delta^4)\sin(wR)}{(k^2+\delta^2)((k-2lw)^2+\delta^2)}f(k)f(2lw-k)\right],
\end{aligned} \tag{4.15}$$

where $\delta = \alpha^2 l$.

Though these expressions are well behaved over the entire ranges of R, w, and $T = (4\pi l)^{-1}$, the physics they embody is not similarly valid. When the system size $2R$ is smaller than a thermal coherence length $\hbar v/k_B T$ ($4\pi l$ in our units), we expect the physics of the three dimensional leads to which the sample is attached to begin playing a role [1]. Moreover the limit in which the Kubo formula for G was derived presumes $Rw < 1$. So taking wR small, and hence wl small, the above expressions at lowest order reduce to

$$\begin{aligned} \mathrm{Re}G(w) &= \frac{e^2}{2h}\left[\int_{-\infty}^{\infty} dk \frac{k^2}{k^2+X^4}\frac{e^k}{(1+e^k)^2} + \pi wl \int_{-\infty}^{\infty} \frac{k^2-X^4}{k^2+X^4} e^k \frac{e^k-1}{(1+e^k)^3}\right], \\ \mathrm{Im}G(w) &= \frac{e^2 wR}{12h}, \end{aligned} \tag{4.16}$$

where $X^2 = 2\pi\delta$. The dc limit of this agrees with the calculation of the d-c conductance in [1].

5. Acknowledgements

The author would like to thank A. LeClair for helpful discussions.

Appendix A. Calculation of the Partition Function in the Limit $R \gg 2l$

We are interested in calculating the sum

$$S_\pm = 2\pi l \sum_k k + \sum_k \log(1 \pm e^{-4\pi lk}), \tag{A.1}$$

where the sum is over k's satisfying

$$1 + X(k) = 1 + e^{-2iRk} \frac{(\alpha^2 - ik)^2}{(\alpha^2 + ik)^2} = 0. \tag{A.2}$$

This sum may be recast as an integral

$$S_\pm = \frac{1}{2\pi i} \int_C dk \frac{X'(k)}{1 + X(k)} \left[2\pi lk + \log(1 \pm e^{-4\pi lk})\right], \tag{A.3}$$

where the contour C is pictured below in Figure A1.

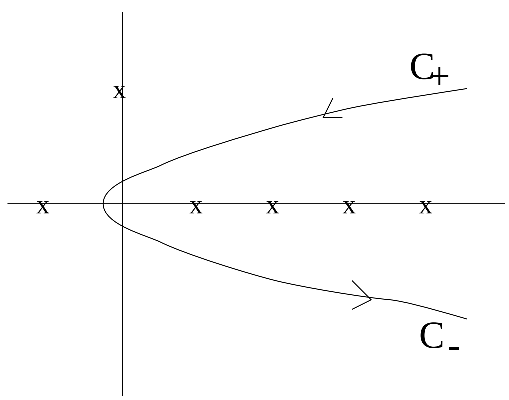

Figure A1 $C = C_- + C_+$

Because $X'(-k) = X'(k)/X^2(k)$ and $X(-k) = X(k)^{-1}$, we can write

$$\int_{C_+} dk \frac{X'}{1+X} \left[2\pi lk + \log(1 \pm e^{-4\pi lk}))\right] = \int_{C_+} dk \frac{X'}{X} \left[2\pi lk + \log(1 \pm e^{-4\pi lk})\right] + \int_{-C_+} dk \frac{X'}{1+X} \left[-2\pi lk + \log(1 \pm e^{4\pi lk})\right], \tag{A.4}$$

where the contour $-C_+$ is pictured in Figure A2.

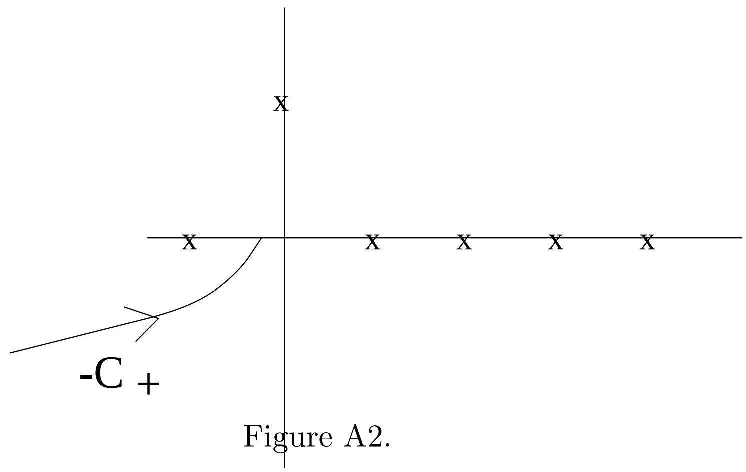

Figure A2.

The second integral on the r.h.s. of the equation can be rewritten as

$$
\begin{aligned}
&\int_{-C_+} dk \frac{X}{1+X}\left[-2\pi lk + \log(1 \pm e^{4\pi lk})\right] = \\
&\qquad \int_{-C_+} dk \frac{X}{1+X}\left[2\pi lk + \log(1 \pm e^{-4\pi lk})\right] + \frac{(1\pm 1)}{2}\pi i \log(2).
\end{aligned}
\tag{A.5}
$$

The sum now reduces to

$$
\begin{aligned}
S_\pm = &\frac{1}{2\pi i}\int_{C_-} dk \frac{X'}{1+X}\left[2\pi lk + \log(1 \pm e^{-4\pi lk})\right] \\
&+ \frac{1}{2\pi i}\int_{C_+} dk \frac{X'}{X}\left[2\pi lk + \log(1 \pm e^{-4\pi lk})\right] \\
&+ \frac{1}{2\pi i}\int_{-C_+} dk \frac{X'}{1+X}\left[2\pi lk + \log(1 \pm e^{4\pi lk})\right] + \frac{(1\pm 1)}{4}\log(2).
\end{aligned}
\tag{A.6}
$$

Combining the first and third integrals we obtain

$$
\begin{aligned}
&\frac{1}{2\pi i}\int_{C_-} dk \frac{X'}{1+X}\left[2\pi lk + \log(1 \pm e^{-4\pi lk})\right] + \frac{1}{2\pi i}\int_{-C_+} dk \frac{X'}{1+X}\left[2\pi lk + \log(1 \pm e^{-4\pi lk})\right] \\
&\qquad = \frac{1}{2\pi i}\int_{C_- + (-C_+)} dk \frac{X'}{1+X}\log(1 \pm e^{-4\pi lk}) \\
&\qquad = \frac{1}{2\pi i}\int_{C_o} dk \frac{\pm 4\pi l e^{-4\pi lk}}{1 \pm e^{-4\pi lk}}\log(1+X) \\
&\qquad = \begin{cases} \sum_{k=1/2,3/2,\cdots} \log(1 + X(-ik/2l)), \\ \sum_{k=1,2,\cdots} \log(1 + X(-ik/2l)), \end{cases} ,
\end{aligned}
\tag{A.7}
$$

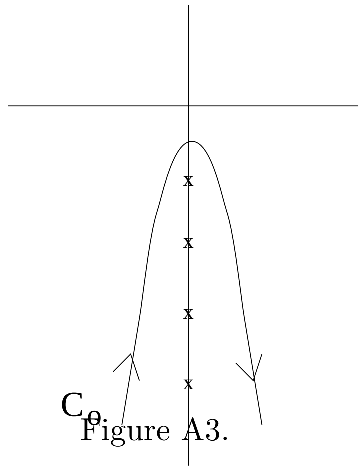

Figure A3.

where in the second to last line we have integrated by parts and deformed the contour to C_o as defined in Figure A3. As $R \to \infty$ this last integral goes to zero. It thus does not contribute to the boundary entropy. So $S_\pm$ reduces to

$$\begin{aligned}\lim_{R\to\infty} S_\pm = -\frac{1}{2\pi}\int_{C_+} dk \left[2R + \frac{4\alpha^2}{\alpha^4+k^2}\right]\left[2\pi lk + \log(1 \pm e^{-4\pi lk})\right] \\ + \frac{1\mp 1}{4}\log(2).\end{aligned} \tag{A.8}$$

The terms proportional to $2\pi lk$ arise from the ground state energy $-1/2\sum k$, and so do not contribute to g. Nor do the terms proportional to R contribute (g is solely a function of boundary length). So the contribution to g from $S_\pm$ is

$$\frac{1}{\pi}\int_0^\infty dk \frac{1}{1+k^2}\log(1\pm e^{-2\pi ak}) + \frac{1\mp 1}{8}\log(2), \tag{A.9}$$

where $a = 2l\alpha^2$.

Appendix B. Calculation of the $j_l - j_r$ Correlators

We wish to evaluate

$$\langle j_{r/l}(x) j_{l/r}(x') \rangle (w) = \frac{1}{16\pi^2 l^2} \left[\int_0^\beta d\tau e^{iw_n \tau} I_\pm \right]_{w_n = -iw + \epsilon} \tag{B.1}$$

where

$$\begin{aligned} I_\pm &= \sum_{k,k' \in Z^+ - 1/2} \frac{kk' - \delta^2}{(k+\delta)(k'+\delta)} e^{-(k+k')(x+x')/2l} e^{\pm i\tau(k+k'-1)/2l}; \\ \beta &= 4\pi l; \\ \delta &= \alpha^2 l. \end{aligned} \tag{B.2}$$

The sums $I_\pm$ can be written as contour integrals

$$I_\pm = -\int_C dk \int_C dk' F_\pm(k,k') g(k,k') e^{\pm(k+k')\tau/2l} \tag{B.3}$$

where

$$\begin{aligned} F_\pm(k,k') &= (1 - f(\mp k))(1 - f(\mp k')); \\ f(k) &= (1 + e^{2\pi k})^{-1}; \\ g(k,k') &= \frac{-kk' - \delta^2}{(-ik+\delta)(-ik'+\delta)} e^{i(k+k')(x+x')/2l}; \end{aligned} \tag{B.4}$$

and the contour C is given below in Figure B1.

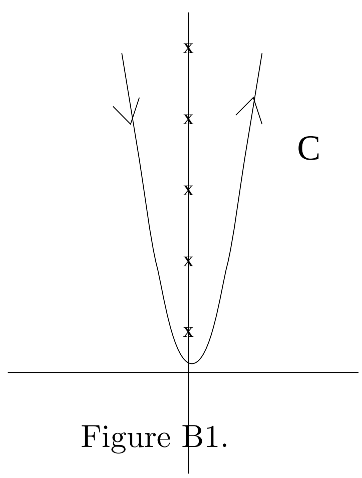

Figure B1.

The Matsubara decomposition of $I_\pm$ equals

$$I_\pm(w_n) = -2l \int_C dk dk' F_\pm(k,k') g(k,k') \frac{e^{\pm(k+k')2\pi} - 1}{i2lw_n \pm (k+k')}, \tag{B.5}$$

where $w_n = n/2l$, $n \in Z$. Because we are going to make the analytic continuation $w_n \to -iw + \epsilon$ we can assume $w_n > 0$. In this case I_+ is identically zero and I_- may be rewritten as

$$I_-(w_n > 0) = -2l \int_C dk dk' F_\pm(k,k') g(k,k') (e^{\pm(k+k')2\pi} - 1) \times \\ \left[\frac{1}{i2lw_n + (k+k')} + \frac{1}{i2lw_n - (k+k')} \right]. \tag{B.6}$$

With I_- in this form, the contours C can be continued to the real axis and the analytic continuation made:

$$I_-(w) = -2l \int_{-\infty}^{\infty} dk dk' F_\pm(k,k') g(k,k') (e^{\pm(k+k')2\pi} - 1) \times \\ \left[\frac{1}{2lw + i\epsilon + (k+k')} + \frac{1}{2lw + i\epsilon - (k+k')} \right]. \tag{B.7}$$

Having made the analytic continuation, we deform the contours back to C, taking into account the pole at $k = w + i\epsilon - k'$:

$$I_-(w) = -2l \int_C dk dk' F_\pm(k,k') g(k,k') (e^{\pm(k+k')2\pi} - 1) \frac{4lw}{4l^2w^2 - (k+k')^2} + \\ 4\pi li (e^{-4\pi wl} - 1) \int_{-\infty}^{\infty} dk F_-(k, 2lw - k) g(k, 2lw - k). \tag{B.8}$$

The integral $\int_C$ vanishes identically because of the presence of $(e^{-2\pi(k+k')} - 1)$. Hence we are left with only the second term. The current-current correlators then reduce to the form claimed:

$$\langle j_r(x) j_l(x') \rangle (w) = 0; \\ \langle j_l(x) j_r(x') \rangle (w) = \frac{i}{4\pi l} (e^{-4\pi wl} - 1) \int_{-\infty}^{\infty} dk F_-(k, 2lw - k) g(k, 2lw - k). \tag{B.9}$$

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